A Mathematical Walkthrough of Weighted Central Moments and Its Relation to Geometric Moment

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ABSTRACT

Moment invariants are a common approach widely used in pattern recognition. The fashionable invariants are proposed by Hu’s who used central moment to generate seven moment invariants. However, these invariants cause a problem with the existence of data which is concentrated near the center-of-mass since some of them are far away from it. Consequently, these data will be neglected, and the calculations are executed for the data which is closed to the center of the mass. Hence, this paper will uncover the mathematical walkthrough of Hu’s moment invariants, their weaknesses, and resolving these limitations by using weighted central moment with Lorentzian function; a function that has the capabilities to provide a proportional portion of an object which is the point concentrated near to the center of mass of the object.

Keywords

Hu’s seven moment invariant, Algebraic Invariant, Lorentzian Function.

Category: I.4.7 [Image Processing and Computer Vision]

1. INTRODUCTION

Since 1960’s, moment and moment function have been extensively employed as image’s invariant global features in pattern recognition. In 1961, Hu was succeeded in generating seven moment invariants. However, later in 2001, Ivar et.al found the weaknesses of Hu’s central moment. This is due to the coordinate of a point of an object which is far away from the center that will be omitted, and the points which are near to the center are kept. This will give inaccurate results if the object contains data that is concentrated near the center-of-mass of the object. Hence, this paper will uncover the weaknesses of Hu’s seven moment invariant mathematically, and the solutions by modifying the conventional invariants with weighted function proposed by Ivar et.al (2001).

2. REVIEW

In 2007, Sergeev et. al had found a fast method for computing Hu’s image moment invariants by generalizing the moments computed in a sliding window with a parallel recursive algorithm. The results were more efficient compared to direct computation. Khaled M Hosny (2008) introduced an exact method that is used for the computation of affine moment invariants for gray level images. The results of the affine moment invariants for symmetric images usually have values equal to zero and this become a problem. Hence, he proposed an accurate computation and automatic selection of normalization parameters to solve this problem (refer to Appendix 1 for substantial reviews of Moment Invariants in Pattern Recognition from 2000 to 2008).

3. MATHEMATICAL MODELING OF HU’S MOMENT INVARIANT

3.1 Moment and Central Moment

Two dimensional \((p + q)^{th}\) order moment of a density function \(\rho(x, y)\) in terms of Riemann integrals is defined as,

\[
m_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q \rho(x, y) dx dy, \quad p, q = 0, 1, 2, \ldots \quad (1)
\]

Assumed that \(\rho(x, y)\) is piecewise continuous, bounded function and only have non zero values in the finite part of the xy plane; then moments of all order exist and the uniqueness theorem is hold[1]. Because we assumed that order all moments exist, then the Moment Generating Function can be expanded into power series in terms of the moment \(m_{pq}\) and the equation becomes,

\[
M(u, v) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} m_{pq} \frac{u^p v^q}{p! q!}.
\]

The Central moments \(\mu_{pq}\) is defined as

\[
\mu_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})^p (y - \bar{y})^q \rho(x, y) dx dy, \quad p, q = 0, 1, 2, \ldots \quad (3)
\]

where

\[
\bar{x} = \frac{m_{10}}{m_{00}}, \quad \bar{y} = \frac{m_{01}}{m_{00}}.
\]
It is well known that under the translation of coordinates, the central moments do not change. So, there is a theorem states that the central moments are invariants under translation. The central moments also can be expressed in terms of the ordinary moments.[1] Therefore, for simplicity, we define moments referred to central moments. And moment generating function \( M(u,v) \) will also be referred to central moments. So the equations become

\[
\begin{align*}
\mu_{pq} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q \rho(x,y) dx dy \\
M(u,v) &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \mu_{pq} \frac{u^p v^q}{p! q!}.
\end{align*}
\]

3.2 Algebraic and Moment Invariant

3.2.1 Algebraic Invariant

A binary algebraic form, or simply a binary form of order \( p \) is a homogeneous polynomial of two variables \( u \) and \( v \) which has form

\[
f = a_0 u^p + \frac{p!}{1!} a_1 u^{p-1} v + \ldots + a_p v^p.
\]

By using Cayley’s notation, the above form may be written as,

\[
f = (a_{ij}; a_{i,0}, \ldots, a_{ij})(u,v)^p = (ua_i + va_j)^p
\]

with

\[
a_{ij} = a_{ji}, i, j \in \mathbb{R} \text{ and } i + j \leq 2
\]

Algebraic invariant is a homogeneous polynomial \( I(a) \) of the coefficients \( a_{ij} \) of weight \( w \) if

\[
I(a_{p0}, \ldots; a_{0p}) = \Delta^w I(a_{ij}),
\]

where \((a_{p0}, \ldots; a_{0p})\) are new coefficients obtained from substituting the following general linear transformation into the original form,

\[
\begin{pmatrix}
u \\
\end{pmatrix} = \begin{pmatrix}
\alpha & \gamma \\
\beta & \delta
\end{pmatrix} \\
\begin{pmatrix}
u \\
\end{pmatrix}, \Delta = \begin{pmatrix}
\alpha & \gamma \\
\beta & \delta
\end{pmatrix} \neq 0.
\]

If \( w = 0 \), the invariance is called an absolute invariant, and if \( w \neq 0 \), it is called relative invariant.

More information about Algebraic Invariant can be seen in [2].

3.2.2 Moment Invariant

From the fact that binary forms are linear in their coefficients, and also assumption that moments are referred to central moments, we can rewrite the equation as

\[
M(u,v) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p! q!} \mu_{pq} (u, v)^p.
\]

This form of moment generating function has a same form as homogenous polynomial. So Hu[1] has declared a moment invariant theorem and has been revised by Reiss with his theorem [3] as follows,

**Theorem** If the binary form of order \( p \) has an algebraic invariant of weight \( w \) and order \( k \),

\[
I(\tilde{a}_{p0}, \ldots; \tilde{a}_{0p}) = \Delta^w I(a_{p0}, \ldots; a_{0p})
\]

then the moment invariant of order \( p \) has the same invariant but with additional factor \( \Delta^w \)

\[
I(\tilde{a}_{p0}, \ldots; \tilde{a}_{0p}) = \Delta^w I(\mu_{p0}, \ldots; \mu_{0p}).
\]

3.2.3 Hu’s Seven Moment Invariant

Based on moment invariant theorem, Hu generated seven moment Invariants as given in [1]. However, many researchers have found the limitations of these invariants as discussed in the next section.

3.3 Weaknesses of Hu’s Moment Invariant

One of the classification problems using central moment occurs if the coordinate of the shape is concentrated near the center of the object. By definition, the central moment can be written as

\[
\mu_{pq} = \sum_{i,j}(x-x_0)^i (y-y_0)^j.
\]

where

\[
(x, y) = \text{the object point of summation,}
\]

\[
(x_0, y_0) = \text{the center of the mass.}
\]

From the above equation, it is clearly seen that if the distance between \( X \) and \( X_0 \) and between \( y \) and \( y_0 \) are too far, i.e., the area region which are too far from the center of the mass, then it will destroy the data or essential information from regions close to the center-of-mass.

If the distance between \( X \) and \( X_0 \) and between \( y \) and \( y_0 \) are big, we can say that

- \((X - X_0)\) goes to infinity (bigger), or \( X \rightarrow \infty \)
- \((y - y_0)\) goes to infinity (bigger), or \( y \rightarrow \infty \)

Hence,

\[
\lim_{i,j}(x-x_0)^i (y-y_0)^j = \sum_{i,j}(x-x_0)^i (y-y_0)^j = \infty
\]

It means that the value of the coordinates which are too far from central moment are bigger and vanish the essential value from coordinates which are near with the center of the mass.
3.4 Weighted Function

Ivar et al. [4] proposed Weighted Central Moment by using Lorentzian function as a weighted function.

3.4.1 Definition of Lorentzian Function

A Lorentz function is defined [5] as,

\[
\frac{1}{1 + a(x-x_0)^2}. \tag{14}
\]

This is a simplified form of the infinite series inverse \(1 + \sum a_i z_i^2\).

For practical purposes, the shortened Lorentz function is accurate enough. The Lorentz function equals the derivative of the arctangent [5].

3.4.2 Modified Lorentzian function

The definition of Lorentzian function as above is just for one variable. In this case, we will use a Lorentzian function for two variables with modification.

Let,

\[
z = x - x_0 \text{ with } x, x_0 \in \mathbb{R}.
\]

Substitute the value of \(z\) to the above equation, we get

\[
\frac{1}{1 + a(x-x_0)^2}. \tag{15}
\]

If we have two variables, then the equation becomes,

\[
\frac{1}{1 + a((x-x_0)^2 + (y-y_0)^2)}. \tag{16}
\]

The utility of Lorentzian is as a weight function in a calculation of moment. Hence, the power of each coordinate will depend on the moment order. Therefore, the modified Lorentzian function becomes

\[
F(x,y) = \frac{1}{1 + \alpha((x-x_0)^2 + (y-y_0)^2)}. \tag{17}
\]

3.4.3 The Weighted Central Moment

The equation of weighted central moment can be written as,

\[
m^*_{pq} = \sum_{x,y} F(x,y) \left[ x - \frac{m_{p0}}{m_{00}} \right]^p \left[ y - \frac{m_{01}}{m_{00}} \right]^q \tag{18}
\]

with

\[
m^*_{pq} = \sum_{x,y} F(x,y)x^p y^q
\]

\[
F(x,y) = \frac{1}{1 + \alpha((x-x_0)^2 + (y-y_0)^2)}
\]

= weighted function

\[
m^*_0 = x_0 = \text{the center of the mass}
\]

\[
m^*_0 = y_0 = \text{the center of the mass}.
\]

4. ANALYSIS

If the distance between a point \((x, y)\) in a shape is far from central point \((x_0, y_0)\), it means that the distance between \((x, y)\) and \((x_0, y_0)\) bigger or in other word it goes to infinity (but not equivalent to infinity), then

\[
\lim_{x\to\infty} F(x,y) = \lim_{y\to\infty} \frac{1}{1 + \alpha((x-x_0)^2 + (y-y_0)^2)} = 0. \tag{19}
\]

Equation (19) illustrate that if the distance between \((x, y)\) and \((x_0, y_0)\) is bigger, then the value of \(F\) will go to zero. Consequently, this point will be omitted from the calculation.

Otherwise, if the distance between a point \((x, y)\) in a shape is far from central point \((x_0, y_0)\), then the distance between \((x, y)\) and \((x_0, y_0)\) is bigger or it goes to infinity (but not equivalent to infinity),

\[
\lim_{x\to\infty} F(x,y) = \lim_{y\to\infty} \frac{1}{1 + \alpha((x-x_0)^2 + (y-y_0)^2)} = \infty. \tag{20}
\]

The above equation illustrates that if the distance between \((x, y)\) and \((x_0, y_0)\) is closer, then the value of \(F\) will bigger. Hence, the point will give a big affect to calculation of central moment and invariants.

5. CONCLUSION

This paper presents a mathematical walkthrough of moment functions and weighted function by using Lorentzian function. Lorentzian function provides a proportional portion for each point in an object, especially for an object which is the point concentrated near to the center – of-mass of the object. A point which is near to the center of the mass will affect the calculation of a moment of an object. Otherwise, the point which is far away to the center will be omitted.

6. ACKNOWLEDGMENTS

This work is supported by Ministry of Higher Education (MOHE) under Fundamental Research Grant Scheme (FRGS VOT 78182).

REFERENCES

Appendix 1

Table 1. List of Publication about Moment Invariant in Year 2000

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invariant Representation and Matching of Space Curves</td>
<td>Lo et.al</td>
<td>Two invariant representations for space curves are discussed in this paper. One represents space curves by complex waveforms. The other represents space curves using the 3-D moment invariants of the data points on the curves. Space curve matching using invariant global features is discussed.</td>
</tr>
<tr>
<td>Weighted Central Moments in Pattern Recognition</td>
<td>Balslev et.al</td>
<td>Discuss the advantages of weighted moments in the determination the rotation angle.</td>
</tr>
</tbody>
</table>

Table 2. List Review of Publication About Moment Invariant in Year 2001

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Neural Network Based Face Recognition with Moment Invariants.</td>
<td>Haddadnia et al.</td>
<td>This paper introduced an experimental evolution of the effectiveness of utilizing various moments as pattern features in human face technology.</td>
</tr>
</tbody>
</table>

Table 3. List of Publication about Moment Invariant in Year 2002

<table>
<thead>
<tr>
<th>Title</th>
<th>Author</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>Blur and Affine Moment Invariant</td>
<td>Tomas Suk, Jan Flusser</td>
<td>The paper is devoted to the recognition of objects and pattern deformed by imaging geometry as well as by un-known blurring.</td>
</tr>
<tr>
<td>Pattern Recognition using Information Slicing Method(PRISM)</td>
<td>Samer Singh, Antony Galton</td>
<td>A method of partitioning feature space of given data into a number of hypercube’s separability measure and the number of elements present in them.</td>
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</tbody>
</table>

Table 4. List of Publication about Moment Invariant in Year 2003

<table>
<thead>
<tr>
<th>Title</th>
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<tbody>
<tr>
<td>Biomimetic (Topological) Pattern Recognition- A new Model of Pattern Recognition Theory and Its Application</td>
<td>Wang Shou-jue, Chen Xu</td>
<td>A new theoretical model of Pattern Recognition which is based on “matter cognition”. The fundamental idea is based on the fact of the continuity in the feature space of any one of the certain kinds of samples.</td>
</tr>
<tr>
<td>United Moment Invariants for Shape Discrimination</td>
<td>Yinan et. al</td>
<td>Proposes some new formulas of moment invariants that are defined united moments.</td>
</tr>
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Table 5. List of Publication about Moment Invariant in Year 2004

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<tr>
<th>Title</th>
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<tbody>
<tr>
<td>Rotationally Invariant Filter Bank for Pattern Recognition</td>
<td>S. Rodtook, S.S. Makhanov</td>
<td>Purposes new rotation moment invariants based on Multi resolution filter bank techniques</td>
</tr>
<tr>
<td>Blurred Image Recognition Based on Complex Moment Invariants</td>
<td>Zhang Tianxu, Liu Jin</td>
<td>Introduced a useful subset of moment invariants that are not affected by the blur, rotation, scale, and translation of the images</td>
</tr>
<tr>
<td>Radikal Zernike Moment Invariant</td>
<td>Belkasim et. al.</td>
<td>New method to solve the drawback of regular Zernike Moment</td>
</tr>
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### Table 6. List of Publication about Moment Invariant in Year 2005

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<tr>
<th>Title</th>
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</thead>
<tbody>
<tr>
<td>Analysis of Moment Invariant’s Stability to Gray-Level Adjustment.</td>
<td>Xingbin Bian, Qingxin Zhu</td>
<td>This paper analyses the relation between invariant moments and gray-level adjustment, and concludes that gray-level histogram equalization can enhance the stability of invariant moments effectively.</td>
</tr>
<tr>
<td>The Curve-Structure Invariant Moments for Shape Analysis and Recognition</td>
<td>Li et. al.</td>
<td>This paper introduces curve structure moment invariants based on the geometric moment invariants from transforming the density in geometric moments into a new density.</td>
</tr>
<tr>
<td>Real Object Recognition Using Moment Invariant</td>
<td>Muharem Murcimex</td>
<td>In this paper, a flexible recognition system that can compute the good features for high classification of 3-D real objects is investigated.</td>
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### Table 7. List of Publication about Moment Invariant in Year 2006

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<tr>
<th>Title</th>
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<tbody>
<tr>
<td>A Fast Discrete Moment Invariant Algorithm and Its Application on Pattern Recognition</td>
<td>Li et. al.</td>
<td>This paper introduces an algorithm that recombines the original moment invariant and the contour moment invariant, which is called a relative contour moment invariant.</td>
</tr>
<tr>
<td>Moment Invariants of Open Curve for Fusion</td>
<td>Zhou-bao, Liao Zhao Liu</td>
<td>This paper presents a new set of moment invariants used for the fusion of open curve target.</td>
</tr>
<tr>
<td>3-D Surface Moment Invariants</td>
<td>Dong Xu, Hua Li</td>
<td>The paper shows that 3-D surface moments, surface moment invariants under similarity transformation and moment invariants can handle the situation where the object is unclosed.</td>
</tr>
<tr>
<td>A Matrix-Based Approach to the Image Moment Problem</td>
<td>Judit Martinez et.al</td>
<td>This paper presents a reconstruction method of an image from its moments that sheds new light on this inverse problem.</td>
</tr>
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### Table 8. List of Publication about Moment Invariant in Year 2007

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<tr>
<th>Title</th>
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<tbody>
<tr>
<td>An Approximation-Based Approach to Computing Image Moment Invariants</td>
<td>V.V.Sergeev, O.A.Titova</td>
<td>A fast method for computing Hu’s image moment invariants is described in this paper.</td>
</tr>
<tr>
<td>An automatic method for generating affine moment invariants</td>
<td>Liu et. al.</td>
<td>In this paper, the notion of generating function is introduced as a simple and straightforward way to derive various affine invariants. Then, we can get the explicit construction of much more affine moment invariants. So, a large set of invariant polynomials can be generated automatically and immediately by the algorithm that was explained in this paper.</td>
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### Table 9. List of Publication about Moment Invariant in Year 2008

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<tr>
<th>Title</th>
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<tbody>
<tr>
<td>On the Computational Aspects of Affine Moment Invariants for Gray-Scale Images.</td>
<td>Khalid M. Hosny</td>
<td>This paper introduce an exact method that is used for the computation of affine moment invariants for gray level images, where the proposed method completely removes the approximation errors.</td>
</tr>
</tbody>
</table>